Turbulent boundary layer equations

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Abstract

We study a boundary layer problem for the Navier-Stokes-alpha model obtaining a generalization of the Prandtl equations conjectured to represent the averaged flow in a turbulent boundary layer. We solve the equations for the semi-infinite plate, both theoretically and numerically. The latter solutions agree with some experimental data in the turbulent boundary layer.

1 Introduction

Boundary-layer theory, first introduced by Ludwig Prandtl in 1904, is now fundamental to many applications of fluid mechanics, especially aerodynamics.

Consider the case of two-dimensional steady incompressible viscous flow near a flat surface. Let x be the coordinate along the horizontal surface, y be the coordinate normal to the surface, and (u, v) be the corresponding velocity of the flow. Near the wall u is significantly larger than v. Also, u changes in y much faster than it does in the x coordinate. Then, in the case of a zero pressure gradient, the following Prandtl equations are used to approximate the Navier-Stokes equations.

$$\begin{cases} u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u = \nu\frac{\partial^2}{\partial y^2}u \\ \frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0, \end{cases} \tag{1}$$

where ν is a kinematic viscosity; density is chosen to be identically one.

A classical example of the boundary layer is a flow near the semi-infinite plate $\{(x,y) : x \geq 0, y = 0\}$. In 1908, Blasius discovered that in this case there exists a similarity variable $\xi = \frac{y}{\sqrt{x}}$. Equations (1) can be reduced to an ordinary differential equation

$$h''' + \frac{1}{2}hh'' = 0 \tag{2}$$

with $h(0) = h'(0) = 0$, and $h'(\xi) \to 1$ as $\xi \to \infty$. Then

$$u(x,y) = u_e h'\left(\frac{y}{\sqrt{l_* x}}\right), \quad v(x,y) = \frac{u_e}{\sqrt{R_x}} h'\left(\frac{y}{\sqrt{l_* x}}\right)$$

are solutions of (1) and they match experimental data in the laminar boundary layer. Here u_e is the horizontal velocity component of the external flow, $l_* = \frac{\nu}{u_0}$, and $R_x = \frac{x}{l_*}$.

For high local Reynolds numbers R_x the flow becomes turbulent and analogs of the equations (1) and (2) were not known. In this paper we use Navier-Stokes-alpha model of fluid turbulence also known as viscous Camassa-Holm equations to study averaged flow, and obtain turbulent boundary layer equations that generalize (1) and (2). Our reduction to an ordinary differential equation (see (5)) also uses Blasius's similarity variable.

Alpha-model was first introduced in [4] as a generalization to n dimensions of the one-dimensional Camassa-Holm equation that describes shallow water waves. Later, NS-α model was proposed as a closure approximation for the Reynolds equations, and its solutions were compared with empirical data for turbulent flows in channels and pipes [1]-[3].

2 Derivation

We study two-dimensional steady incompressible viscous flow near a flat surface. Let x be the coordinate along the surface, y be the coordinate normal to the surface. Denote also (u,v) to be the velocity of the flow.

2-D Navier-Stokes-α model is used to study a boundary layer flow.

$$\begin{cases} \frac{\partial}{\partial t}\mathbf{v} + (\mathbf{u}\cdot\nabla)\mathbf{v} + v_j \nabla u_j = \nu\Delta\mathbf{v} - \nabla q \\ \nabla\cdot\mathbf{u} = 0, \end{cases} \tag{3}$$

where $\mathbf{u} = (u,v)$, and $\mathbf{v} = (\gamma,\tau)$ is a momentum defined in the following way:

$$\mathbf{v} = \mathbf{u} - \frac{\partial}{\partial x_i}\left(\alpha^2 \delta_{ij} \frac{\partial}{\partial x_j}\mathbf{u}\right).$$

Non-slip boundary conditions are used: $\mathbf{u}|_{y=0} = 0$. The other boundary conditions are going to be determined later.

Fix l on the x-axis and define $\epsilon(l)$ to be $\epsilon := 1/\sqrt{R_l} = \sqrt{\nu/(u_e l)}$, where ν is viscosity and u_e is the horizontal velocity component of the external flow.

We change variables $x_1 = \frac{1}{l}x$, $y_1 = \frac{1}{\epsilon l}y$, $u_1 = \frac{1}{u_e}u$, $v_1 = \frac{1}{\epsilon u_e}v$, $q_1 = \frac{1}{u_e^2}q$. $\alpha_1 = \frac{\alpha}{\epsilon l}$. In addition, assume that α_1 doesn't depend on y variable. After neglecting terms with high powers of ϵ, dropping subscripts and denoting

$$w = \left(1 - \alpha^2 \frac{\partial^2}{\partial y^2}\right) u,$$

we derive the following turbulent boundary layer equations that generalize Prandtl equations:

$$\begin{cases} u\frac{\partial}{\partial x}w + v\frac{\partial}{\partial y}w + w\frac{\partial}{\partial x}u = \frac{\partial^2}{\partial y^2}w - \frac{\partial}{\partial x}q \\ w\frac{\partial}{\partial y}u = -\frac{\partial}{\partial y}q \\ \frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0. \end{cases} \tag{4}$$

Note that derivative of q is not zero in y-direction. Therefore, we introduce $Q(x,y) := q + \frac{1}{2}u^2 - \frac{1}{2}\alpha^2\left(\frac{\partial}{\partial y}u\right)^2$, and obtain the following equations:

$$\begin{cases} u\frac{\partial}{\partial x}w + v\frac{\partial}{\partial y}w + \alpha^2\left(\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2}\right) + \frac{1}{2}\frac{\partial}{\partial x}\alpha^2 \cdot \left(\frac{\partial}{\partial y}u\right)^2 = \frac{\partial^2}{\partial y^2}w - \frac{\partial}{\partial x}Q \\ \frac{\partial}{\partial y}Q = 0 \\ \frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0. \end{cases}$$

Consider now a case of a two-dimensional steady incompressible viscous flow near the semi-infinite plate $\{(x,y) : x \geq 0, y = 0\}$. The following assumptions are made in this case:

- $\alpha = \sqrt{x}\beta$
- Zero pressure gradient, i.e. $\frac{\partial}{\partial x}Q = 0$.

In addition, we will study the solutions (u_∞, v_∞) of (4), which on some adequate interval $x_1 \leq x \leq x_2$ are of the form

$$u_\infty = f(\xi),\ v_\infty = \frac{1}{\sqrt{x}}g(\xi),\ \xi = \frac{y}{\sqrt{x}}.$$

Let $h(\xi) = \int_0^\xi f(\eta)d\eta$. Then $g = \frac{1}{2}\xi h' - \frac{1}{2}h$, and we have the following equation for h:

$$h''' + \frac{1}{2}hh'' - \beta^2\left(h''''' + \frac{1}{2}hh''''\right) = 0. \tag{5}$$

The boundary conditions are $h(0) = h'(0) = 0$, $h''(0) > 0$, and $h'(\xi) \to 1$ as $\xi \to \infty$. Note that if $h(\xi)$ is a solution of (5), then $\hat{h}(x) := \beta h(\beta x)$ is a solution of

$$h''' + \frac{1}{2}hh'' - \left(h''''' + \frac{1}{2}hh''''\right) = 0. \tag{6}$$

This equation can be also written as $m''' + \frac{1}{2}hm'' = 0$, where $m = h - h''$. The following theorem is proven for this equation in [6].

Theorem 2.1 *Given any $a > 0$, b, there exists $c(a, b)$ such that $h'(\xi) \to$ const. ≥ 0 as $\xi \to \infty$, where h is a solution of (6) with $h(0) = h'(0) = 0$, $h''(0) = a$, $h'''(0) = b$, $h''''(0) = c$.*

3 Comparison with Experimental Data

It is common to use

$$y^+ = \frac{u_\tau y}{\nu}, \quad u^+ = \frac{u}{u_\tau}$$

in the turbulent boundary layer, where $u_\tau = \sqrt{\nu \frac{\partial u}{\partial y}\Big|_{y=0}}$. Fix x on the horizontal axis and denote $l_* = \frac{\nu}{u_e}$, $R_x = \frac{x}{l_*}$. As argued in the section 2,

$$u = u_e h'\left(\frac{y}{\sqrt{l_* x}}\right)$$

represents a horizontal component of the averaged velocity for some h satisfying (5) with $h(0) = h'(0) = 0$, $h''(0) = a > 0$, $h'''(0) = b$, and $h'(\xi) \to 1$ as $\xi \to \infty$.

For any such h, $\hat{h}(\xi) = \beta h(\beta\xi)$ satisfies (6). Then $\beta^2 = \lim_{\xi\to\infty} \hat{h}'(\xi)$. In addition,

$$u^+ = \frac{R_x^{1/4}}{\sqrt{a}\beta^2}\hat{h}'\left(\frac{y^+}{\sqrt{a}\beta R_x^{1/4}}\right).$$

Given c_f, a skin-friction coefficient and R_θ, a Reynolds number based on momentum thickness we find a, b, and R_x so that the following conditions hold:

1. $c_f = 2/\left(\inf_y u^+\right)^2$.
2. $R_\theta = \frac{1}{\nu}\int_0^\delta u\left(1 - \frac{u}{u_e}\right)dy$.
3. Von Karman log law for the middle inflection point in logarithmic coordinates.

A family of curves $\{u\}_{c_f,R_\theta}$ was compared with experimental data of Rolls-Royce applied science laboratory, ERCOFTAC t3b test case (see Fig. 1, 2, and 3). Comparison shows that the case $a + \beta b > 0$ corresponds to a laminar region of a boundary layer, $a + \beta b < 0$ corresponds to a turbulent region of a boundary layer. The case $a + \beta b = 0$ corresponds to a transition point.

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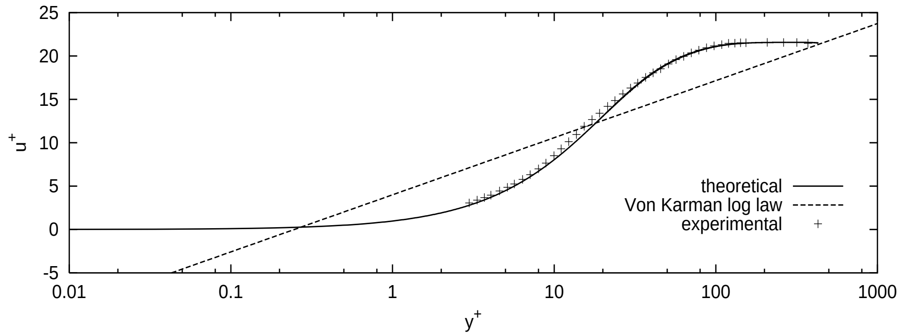

Figure 1: $c_f = 4.32, \; R_\theta = 265, \; \beta = 2.22$

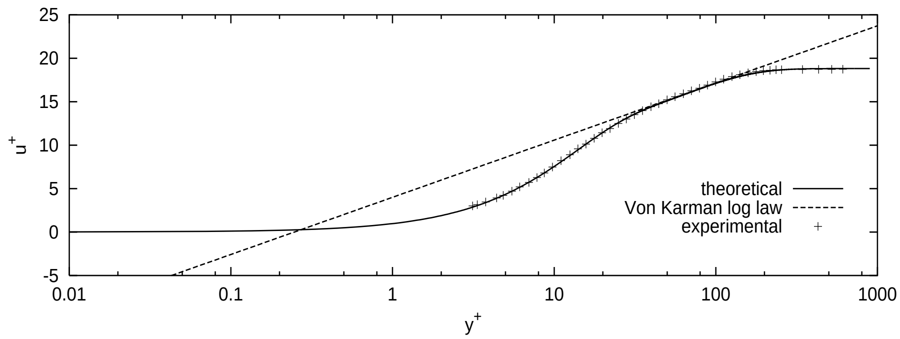

Figure 2: $c_f = 5.69, \; R_\theta = 396, \; \beta = 6.63$

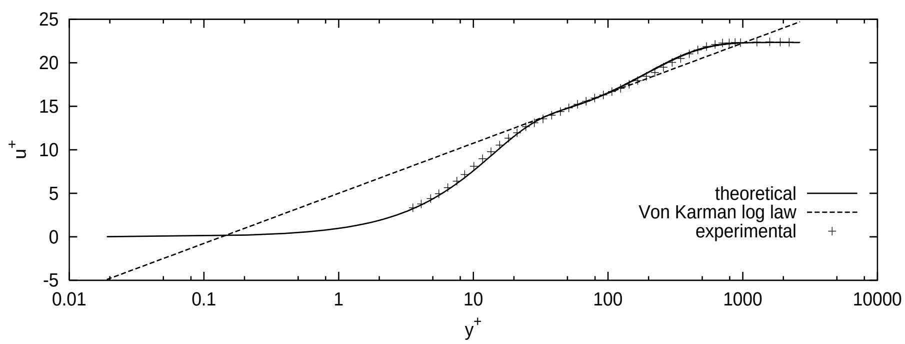

Figure 3: $c_f = 4.01, \; R_\theta = 1436, \; \beta = 19.8$